Contents 8

Matrix Solution of Equations

Learning outcomes

In this Workbook you will learn to apply your knowledge of matrices to solve systems of linear equations. Such systems of equations arise very often in mathematics, science and engineering. Three basic techniques are outlined, Cramer's method, the inverse matrix approach and the Gauss elimination method. The Gauss elimination method is, by far, the most widely used (since it can be applied to all systems of linear equations). However, you will learn that, for certain (usually small) systems of linear equations the other two techniques may be better.

Solution by Cramer's Rule

8.1

Introduction

The need to solve systems of linear equations arises frequently in engineering. The analysis of electric circuits and the control of systems are two examples. Cramer's rule for solving such systems involves the calculation of determinants and their ratio. For systems containing only a few equations it is a useful method of solution.

Prerequisites

Before starting this Section you should ...

- be able to evaluate 2×2 and 3×3 determinants

Learning Outcomes

On completion you should be able to ...

- state and apply Cramer's rule to find the solution of two simultaneous linear equations
- state and apply Cramer's rule to find the solution of three simultaneous linear equations
- recognise cases where the solution is not unique or a solution does not exist

1. Solving two equations in two unknowns

If we have one linear equation

$$ax = b$$

in which the unknown is x and a and b are constants then there are just three possibilities:

- $a \neq 0$ then $x = \dfrac{b}{a} = a^{-1}b$. In this case the equation $ax = b$ has a **unique solution** for x.
- $a = 0$, $b = 0$ then the equation $ax = b$ becomes $0 = 0$ and any value of x will do. In this case the equation $ax = b$ has **infinitely many solutions**.
- $a = 0$ and $b \neq 0$ then $ax = b$ becomes $0 = b$ which is a contradiction. In this case the equation $ax = b$ has **no solution** for x.

What happens if we have more than one equation and more than one unknown? We shall find that the solutions to such systems can be characterised in a manner similar to that occurring for a single equation; that is, a system may have a unique solution, an infinity of solutions or no solution at all. In this Section we examine a method, known as Cramer's rule and employing determinants, for solving systems of linear equations.
Consider the equations

$$ax + by = e \qquad (1)$$

$$cx + dy = f \qquad (2)$$

where a, b, c, d, e, f are given numbers. The variables x and y are unknowns we wish to find. The pairs of values of x and y which **simultaneously** satisfy both equations are called solutions. Simple algebra will eliminate the variable y between these equations. We multiply equation (1) by d, equation (2) by b and subtract:

first, $(1) \times d$ $\qquad adx + bdy = ed$

then, $(2) \times b$ $\qquad bcx + bdy = bf$

(we multiplied in this way to make the coefficients of y equal.)
Now subtract to obtain

$$(ad - bc)x = ed - bf \qquad (3)$$

Task

Starting with equations (1) and (2) above, eliminate x.

Your solution

Answer

Multiply equation (1) by c and equation (2) by a to obtain

$$acx + bcy = ec \qquad \text{and} \qquad acx + ady = af.$$

Now subtract to obtain

$$(bc - ad)y = ec - af$$

If we multiply this last equation in the Task above by -1 we obtain

$$(ad - bc)y = af - ec \tag{4}$$

Dividing equations (3) and (4) by $ad - bc$ we obtain the solutions

$$x = \frac{ed - bf}{ad - bc}\ , \quad y = \frac{af - ec}{ad - bc} \tag{5}$$

There is of course one proviso: if $ad - bc = 0$ then neither x nor y has a defined value.

If we choose to express these solutions in terms of determinants we have the formulation for the solution of simultaneous equations known as **Cramer's rule**.

If we define Δ as the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and provided $\Delta \neq 0$ then the **unique** solution of the equations

$$ax + by = e$$

$$cx + dy = f$$

is by (5) given by

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta} \quad \text{where} \quad \Delta_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

Now Δ is the determinant of coefficients on the left-hand sides of the equations. In the expression Δ_x the coefficients of x (i.e. $\begin{pmatrix} a \\ c \end{pmatrix}$ which is column 1 of Δ) are replaced by the terms on the right-hand sides of the equations (i.e. by $\begin{pmatrix} e \\ f \end{pmatrix}$). Similarly in Δ_y the coefficients of y (column 2 of Δ) are replaced by the terms on the right-hand sides of the equations.

Key Point 1

Cramer's Rule for Two Equations

The unique solution to the equations:

$$ax + by = e$$
$$cx + dy = f$$

is given by:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}$$

in which

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \Delta_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}, \qquad \Delta_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

If $\Delta = 0$ this method of solution cannot be used.

Task

Use Cramer's rule to solve the simultaneous equations

$$\begin{aligned} 2x + y &= 7 \\ 3x - 4y &= 5 \end{aligned}$$

Your solution

Answer

Calculating $\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11$. Since $\Delta \neq 0$ we can proceed with Cramer's solution.

$$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11 \qquad x = \frac{1}{\Delta}\begin{vmatrix} 7 & 1 \\ 5 & -4 \end{vmatrix}, \quad y = \frac{1}{\Delta}\begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix}$$

i.e. $x = \dfrac{(-28-5)}{(-11)}$, $y = \dfrac{(10-21)}{(-11)}$ implying: $x = \dfrac{-33}{-11} = 3$, $y = \dfrac{-11}{-11} = 1$.

You can check by direct substitution that these are the exact solutions to the equations.

Task

Use Cramer's rule to solve the equations

$$\text{(a)}\quad \begin{aligned} 2x-3y &= 6\\ 4x-6y &= 12 \end{aligned} \qquad\qquad \text{(b)}\quad \begin{aligned} 2x-3y &= 6\\ 4x-6y &= 10 \end{aligned}$$

Your solution

Answer

You should have checked $\begin{vmatrix} 2 & -3 \\ 4 & -6 \end{vmatrix}$ first, since

$\begin{vmatrix} 2 & -3 \\ 4 & -6 \end{vmatrix} = -12-(-12) = 0$. Hence there is no unique solution in either case.

In the system (a) the second equation is twice the first so there are infinitely many solutions. (Here we can give y *any* value we wish, t say; but then the x value is always $(6+3t)/2$. So for each value of t there are values for x and y which simultaneously satisfy both equations. There is an infinite number of possible solutions). In (b) the equations are inconsistent (since the first is $2x-3y=6$ and the second is $2x-3y=5$ which is not possible). Hence there are no solutions.

Notation

For ease of generalisation to larger systems we write the two-equation system in a different notation:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1\\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Here the unknowns are x_1 and x_2, the right-hand sides are b_1 and b_2 and the coefficients are a_{ij} where, for example, a_{21} is the coefficient of x_1 in equation two. In general, a_{ij} is the coefficient of x_j in equation i.

Cramer's rule can then be stated as follows:

If $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$, then the equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1\\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

have solution

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

2. Solving three equations in three unknowns

Cramer's rule can be extended to larger systems of simultaneous equations but the calculational effort increases rapidly as the size of the system increases. We quote Cramer's rule for a system of three equations.

Key Point 2

Cramer's Rule for Three Equations

The unique solution to the system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

is

$$x_1 = \frac{\Delta_{x_1}}{\Delta}, \quad x_2 = \frac{\Delta_{x_2}}{\Delta}, \quad x_3 = \frac{\Delta_{x_3}}{\Delta}$$

in which

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and

$$\Delta_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad \Delta_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \qquad \Delta_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

If $\Delta = 0$ this method of solution cannot be used.

Notice that the structure of the fractions is similar to that for the two-equation case. For example, the determinant forming the numerator of x_1 is obtained from the determinant of coefficients, Δ, by replacing the first column by the right-hand sides of the equations.

Notice too the increase in calculation: in the two-equation case we had to evaluate three 2×2 determinants, whereas in the three-equation case we have to evaluate four 3×3 determinants. Hence Cramer's rule is not really practicable for larger systems.

Use Cramer's rule to solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 5 \\ 3x_1 - x_2 + 2x_3 &= 12. \end{aligned}$$

First check that $\Delta \neq 0$:

Your solution

Answer

$$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix}.$$

Expanding along the top row,

$$\begin{aligned} \Delta &= 1 \times \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ &= 1 \times (2-1) + 2 \times (4+3) + 1 \times (-2-3) \\ &= 1 + 14 - 5 = 10 \end{aligned}$$

Now find the value of x_1. First write down the expression for x_1 in terms of determinants:

Your solution

Answer

$$x_1 = \begin{vmatrix} 3 & -2 & 1 \\ 5 & 1 & -1 \\ 12 & -1 & 2 \end{vmatrix} \div \Delta$$

Now calculate x_1 explicitly:

Your solution

Answer

The numerator is found by expanding along the top row to be

$$\begin{aligned} 3 \quad \times & \left|\begin{array}{rr} 1 & -1 \\ -1 & 2 \end{array}\right| - (-2) \times \left|\begin{array}{rr} 5 & -1 \\ 12 & 2 \end{array}\right| + 1 \times \left|\begin{array}{rr} 5 & 1 \\ 12 & -1 \end{array}\right| \\ = & \ 3 \times 1 + 2 \times 22 + 1 \times (-17) \\ = & \ 30 \end{aligned}$$

Hence $x_1 = \dfrac{1}{10} \times 30 = 3$

In a similar way find the values of x_2 and x_3:

Your solution

Answer

$$\begin{aligned} x_2 &= \frac{1}{10}\left|\begin{array}{rrr} 1 & 3 & 1 \\ 2 & 5 & -1 \\ 3 & 12 & 2 \end{array}\right| \\ &= \frac{1}{10}\left\{1 \times \left|\begin{array}{rr} 5 & -1 \\ 12 & 2 \end{array}\right| - 3 \times \left|\begin{array}{rr} 2 & -1 \\ 3 & 2 \end{array}\right| + 1 \times \left|\begin{array}{rr} 2 & 5 \\ 3 & 12 \end{array}\right|\right\} \\ &= \frac{1}{10}\{22 - 3 \times 7 + 9\} = 1 \\ x_3 &= \frac{1}{10}\left\{1 \times \left|\begin{array}{rr} 1 & 5 \\ -1 & 12 \end{array}\right| - (-2) \times \left|\begin{array}{rr} 2 & 5 \\ 3 & 12 \end{array}\right| + 3 \times \left|\begin{array}{rr} 2 & 1 \\ 3 & -1 \end{array}\right|\right\} \\ &= \frac{1}{10}\{17 + 2 \times 9 + 3 \times (-5)\} = 2 \end{aligned}$$

Engineering Example 1

Stresses and strains on a section of material

Introduction

An important engineering problem is to determine the effect on materials of different types of loading. One way of measuring the effects is through the strain or fractional change in dimensions in the material which can be measured using a strain gauge.

Problem in words

In a homogeneous, isotropic and linearly elastic material, the strains (i.e. fractional displacements) on a section of the material, represented by $\varepsilon_x,\ \varepsilon_y,\ \varepsilon_z$ for the x-, y-, z-directions respectively, can be related to the stresses (i.e. force per unit area), $\sigma_x,\ \sigma_y,\ \sigma_z$ by the following system of equations.

$$\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\ \sigma_x \quad -v\sigma_y \quad -v\sigma_z\ ) \\
\varepsilon_y &= \frac{1}{E}(\ -v\sigma_x \quad +\sigma_y \quad -v\sigma_z\ ) \\
\varepsilon_z &= \frac{1}{E}(\ -v\sigma_x \quad -v\sigma_y \quad +\sigma_z\ )
\end{aligned}$$

where E is the modulus of elasticity (also called Young's modulus) and v is Poisson's ratio which relates the lateral strain to the axial strain.

Find expressions for the stresses $\sigma_x,\ \sigma_y,\ \sigma_z$, in terms of the strains $\varepsilon_x,\ \varepsilon_y$, and ε_z.

Mathematical statement of problem

The given system of equations can be written as a matrix equation:

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -v & -v \\ -v & 1 & -v \\ -v & -v & 1 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

We can write this equation as

$$\varepsilon = \frac{1}{E}\boldsymbol{A\sigma}$$

where $\varepsilon = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} 1 & -v & -v \\ -v & 1 & -v \\ -v & -v & 1 \end{pmatrix}$ and $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$

This matrix equation must be solved to find the vector $\boldsymbol{\sigma}$ in terms of the vector ε and the inverse of the matrix $\boldsymbol{A}$.

Mathematical analysis

$$\varepsilon = \frac{1}{E}\boldsymbol{A\sigma}$$

Multiplying both sides of the expression by E we get

$$E\varepsilon = \boldsymbol{A\sigma}$$

Multiplying both sides by $\boldsymbol{A}^{-1}$ we find that:

$$\boldsymbol{A}^{-1}E\varepsilon = \boldsymbol{A}^{-1}\boldsymbol{A\sigma}$$

But $\boldsymbol{A}^{-1}\boldsymbol{A} = \mathbf{I}$ so this becomes

$$\boldsymbol{\sigma} = E\boldsymbol{A}^{-1}\varepsilon$$

To find expressions for the stresses $\sigma_x,\ \sigma_y,\ \sigma_z$, in terms of the strains $\varepsilon_x,\ \varepsilon_y$ and ε_z, we must find the inverse of the matrix **A**.

To find the inverse of $\begin{pmatrix} 1 & -v & -v \\ -v & 1 & -v \\ -v & -v & 1 \end{pmatrix}$ we first find the matrix of minors which is:

$$\begin{pmatrix} \begin{vmatrix} 1 & -v \\ -v & 1 \end{vmatrix} & \begin{vmatrix} -v & -v \\ -v & 1 \end{vmatrix} & \begin{vmatrix} -v & 1 \\ -v & -v \end{vmatrix} \\ \\ \begin{vmatrix} -v & -v \\ -v & 1 \end{vmatrix} & \begin{vmatrix} 1 & -v \\ -v & 1 \end{vmatrix} & \begin{vmatrix} 1 & -v \\ -v & -v \end{vmatrix} \\ \\ \begin{vmatrix} -v & -v \\ 1 & -v \end{vmatrix} & \begin{vmatrix} 1 & -v \\ -v & -v \end{vmatrix} & \begin{vmatrix} 1 & -v \\ -v & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1-v^2 & -v-v^2 & v^2+v \\ -v-v^2 & 1-v^2 & -v-v^2 \\ v^2+v & -v-v^2 & 1-v^2 \end{pmatrix}.$$

We then apply the pattern of signs:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

to obtain the matrix of cofactors

$$\begin{pmatrix} 1-v^2 & v+v^2 & v^2+v \\ v+v^2 & 1-v^2 & v+v^2 \\ v^2+v & v+v^2 & 1-v^2 \end{pmatrix}.$$

To find the adjoint we take the transpose of the above, (which is the same as the original matrix since the matrix is symmetric)

$$\begin{pmatrix} 1-v^2 & v+v^2 & v^2+v \\ v+v^2 & 1-v^2 & v+v^2 \\ v^2+v & v+v^2 & 1-v^2 \end{pmatrix}.$$

The determinant of the original matrix is

$$1 \times (1-v^2) - v(v+v^2) - v(v^2+v) = 1 - 3v^2 - 2v^3.$$

Finally we divide the adjoint by the determinant to find the inverse, giving

$$\frac{1}{1-3v^2-2v^3}\begin{pmatrix} 1-v^2 & v+v^2 & v+v^2 \\ v+v^2 & 1-v^2 & v+v^2 \\ v+v^2 & v+v^2 & 1-v^2 \end{pmatrix}.$$

Now we found that $\boldsymbol{\sigma} = E\boldsymbol{A}^{-1}\varepsilon$ so $\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} = \dfrac{E}{1-3v^2-2v^3}\begin{pmatrix} 1-v^2 & v+v^2 & v+v^2 \\ v+v^2 & 1-v^2 & v+v^2 \\ v+v^2 & v+v^2 & 1-v^2 \end{pmatrix}\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix}$

We can write this matrix equation as 3 equations relating the stresses $\sigma_x,\ \sigma_y,\ \sigma_z$, in terms of the strains $\varepsilon_x,\ \varepsilon_y$ and ε_z, by multiplying out this matrix expression, giving:

$$\begin{aligned}
\sigma_x &= \frac{E}{1-3v^2-2v^3}\ \big((1-v^2)\varepsilon_x + (v+v^2)\varepsilon_y + (v+v^2)\varepsilon_z\big) \\
\sigma_y &= \frac{E}{1-3v^2-2v^3}\ \big((v+v^2)\varepsilon_x + (1-v^2)\varepsilon_y + (v+v^2)\varepsilon_z\big) \\
\sigma_z &= \frac{E}{1-3v^2-2v^3}\ \big((v+v^2)\varepsilon_x + (v+v^2)\varepsilon_y + (1-v^2)\varepsilon_z\big)
\end{aligned}$$

Interpretation

Matrix manipulation has been used to transform three simultaneous equations relating strain to stress into simultaneous equations relating stress to strain in terms of the elastic constants. These would be useful for deducing the applied stress if the strains are known. The original equations enable calculation of strains if the applied stresses are known.

Exercises

1. Solve the following using Cramer's rule:

(a) $\begin{aligned} 2x - 3y &= 1 \\ 4x + 4y &= 2 \end{aligned}$ (b) $\begin{aligned} 2x - 5y &= 2 \\ -4x + 10y &= 1 \end{aligned}$ (c) $\begin{aligned} 6x - y &= 0 \\ 2x - 4y &= 1 \end{aligned}$

2. Using Cramer's rule obtain the solutions to the following sets of equations:

(a) $\begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ x_1 + x_3 &= 4 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$ (b) $\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$

Answers

1. (a) $x = \dfrac{1}{2},\ y = 0$ (b) $\Delta = 0$, no solution (c) $x = -\dfrac{1}{22},\ y = -\dfrac{3}{11}$

2. (a) $x_1 = \dfrac{8}{3},\ x_2 = -4,\ x_3 = \dfrac{4}{3}$ (b) $x_1 = \dfrac{1}{2},\ x_2 = 1,\ x_3 = \dfrac{3}{2}$

Solution by Inverse Matrix Method

8.2

Introduction

The power of matrix algebra is seen in the representation of a system of simultaneous linear equations as a matrix equation. Matrix algebra allows us to write the solution of the system using the inverse matrix of the coefficients. In practice the method is suitable only for small systems. Its main use is the theoretical insight into such problems which it provides.

Prerequisites

Before starting this Section you should . . .

- be familiar with the basic rules of matrix algebra
- be able to evaluate 2×2 and 3×3 determinants
- be able to find the inverse of 2×2 and 3×3 matrices

Learning Outcomes

On completion you should be able to . . .

- use the inverse matrix of coefficients to solve a system of two linear simultaneous equations
- use the inverse matrix of coefficients to solve a system of three linear simultaneous equations
- recognise and identify cases where there is no solution or no unique solution

1. Solving a system of two equations using the inverse matrix

If we have one linear equation

$$ax = b$$

in which the unknown is x and a and b are constants and $a \neq 0$ then $x = \dfrac{b}{a} = a^{-1}b$.
What happens if we have more than one equation and more than one unknown? In this Section we copy the algebraic solution $x = a^{-1}b$ used for a single equation to solve a system of linear equations. As we shall see, this will be a very natural way of solving the system if it is first written in matrix form.
Consider the system

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 - 2x_2 &= -1. \end{aligned}$$

In matrix form this becomes

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \qquad \text{which is of the form} \qquad AX = B.$$

If A^{-1} exists then the solution is

$$X = A^{-1}B. \qquad \text{(compare the solution } x = a^{-1}b \text{ above)}$$

Task

Given the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ find its determinant. What does this tell you about A^{-1}?

Your solution

Answer

$|A| = 2 \times (-2) - 1 \times 3 = -7$

since $|A| \neq 0$ then A^{-1} exists.

Now find A^{-1}

Your solution

Answer

$$A^{-1} = \frac{1}{(-7)} \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

Task

Solve the system $AX = B$ where $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ and B is $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

Your solution

Answer

$X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence $x_1 = 1,\ x_2 = 1.$

Task

Use the inverse matrix method to solve

$$2x_1 + 3x_2 = 3$$

$$5x_1 + 4x_2 = 11$$

Your solution

Answer

$AX = B$ is $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$

$$|A| = 2 \times 4 - 3 \times 5 = -7 \quad \text{and} \quad A^{-1} = -\frac{1}{7}\begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix}$$

Using $X = A^{-1}B$:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 11 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} 4 \times 3 - 3 \times 11 \\ -5 \times 3 + 2 \times 11 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -21 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

So $x_1 = 3,\ x_2 = -1$

Engineering Example 2

Currents in two loops

In the circuit shown find the currents $(i_1,\ i_2)$ in the loops.

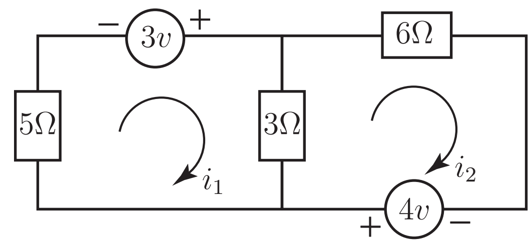

Figure 1

Solution

We note that the current across the $3\ \Omega$ resistor (travelling top to bottom in the diagram) is given by $(i_1 - i_2)$. With this proviso we can apply Kirchhoff's law:

In the left-hand loop $\quad 3(i_1 - i_2) + 5i_1 = 3 \quad \rightarrow \quad 8i_1 - 3i_2 = 3$

In the right-hand loop $\quad 3(i_2 - i_1) + 6i_2 = 4 \quad \rightarrow \quad -3i_1 + 9i_2 = 4$

In matrix form:

$$\begin{bmatrix} 8 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 8 & -3 \\ -3 & 9 \end{bmatrix}$ is $\dfrac{1}{63}\begin{bmatrix} 9 & 3 \\ 3 & 8 \end{bmatrix}$ so solving gives

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \frac{1}{63}\begin{bmatrix} 9 & 3 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{63}\begin{bmatrix} 39 \\ 41 \end{bmatrix}$$

so $\quad i_1 = \dfrac{39}{63} \quad i_2 = \dfrac{41}{63}$

2. Non-unique solutions

The key to obtaining a unique solution of the system $AX = B$ is to find A^{-1}. What happens when A^{-1} does not exist?
Consider the system

$$2x_1 + 3x_2 = 5 \qquad (1)$$

$$4x_1 + 6x_2 = 10 \qquad (2)$$

In matrix form this becomes

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

Task

Identify the matrix A and hence find A^{-1}.

Your solution

Answer

$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ and $|A| = 2 \times 6 - 4 \times 3 = 0$. Hence A^{-1} does not exist.

Looking at the original system we see that Equation (2) is simply Equation (1) with all coefficients doubled. In effect we have only one equation for the two unknowns x_1 and x_2. The equations are **consistent** and have **infinitely many solutions**.

If we let x_2 assume a **particular** value, t say, then x_1 must take the value $x_1 = \dfrac{1}{2}(5 - 3t)$ i.e. the solution to the given equations is:

$$x_2 = t, \quad x_1 = \frac{1}{2}(5 - 3t), \qquad \text{where } t \text{ is called a parameter.}$$

For each value of t there are unique values for x_1 and x_2 which satisfy the original system of equations. For example, if $t = 1$, then $x_2 = 1$, $x_1 = 1$, if $t = -3$ then $x_2 = -3$, $x_1 = 7$ and so on.
Now consider the system

$$2x_1 + 3x_2 = 5 \qquad (3)$$

$$4x_1 + 6x_2 = 9 \qquad (4)$$

Since the left-hand sides are the same as those in the previous system then A is the same and again A^{-1} does not exist. There is **no solution** to the Equations (3) and (4).
However, if we double Equation (3) we obtain

$$4x_1 + 6x_2 = 10,$$

which conflicts with Equation (4). There are thus no solutions to (3) and (4) and the equations are said to be **inconsistent**.

Task

What can you conclude about the solutions of the following systems?

(a) $$\begin{array}{rcrcl} x_1 & - & 2x_2 & = & 1 \\ 3x_1 & - & 6x_2 & = & 3 \end{array}$$ (b) $$\begin{array}{rcrcl} 3x_1 & + & 2x_2 & = & 7 \\ -6x_1 & - & 4x_2 & = & 5 \end{array}$$

First write the systems in matrix form and find $|A|$:

Your solution

(a)

(b)

Answer

(a) $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad |A| = -6 + 6 = 0;$

(b) $\begin{bmatrix} 3 & 2 \\ -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ $\quad |A| = -12 + 12 = 0.$

Now compare the two equations in each system in turn:

Your solution

(a)

(b)

Answer

(a) The second equation is 3 times the first equation. There are infinitely many solutions of the form $x_2 = t,\ x_1 = 1 + 2t$ where t is arbitrary.

(b) If we multiply the first equation by (-2) we obtain $-6x_1 - 4x_2 = -14$ which is in conflict with the second equation. The equations are inconsistent and have no solution.

3. Solving three equations in three unknowns

It is much more tedious to use the inverse matrix to solve a system of three equations although in principle, the method is the same as for two equations.
Consider the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 5 \\ 3x_1 - x_2 + 2x_3 &= 12 \end{aligned}$$

We met this system in Section 8.1 where we found that $|A| = 10$. Hence A^{-1} exists.

Task

Find A^{-1} by the method of determinants.

First form the matrix where each element of A is replaced by its minor:

Your solution

Answer

$$\begin{bmatrix} \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 7 & -5 \\ -3 & -1 & 5 \\ 1 & -3 & 5 \end{bmatrix}.$$

Now use the 3×3 array of signs to obtain the matrix of cofactors:

Your solution

Answer

The array of signs is $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ so that we obtain $\begin{bmatrix} 1 & -7 & -5 \\ 3 & -1 & -5 \\ 1 & 3 & 5 \end{bmatrix}$.

Now transpose this matrix and divide by $|A|$ to obtain A^{-1}:

Your solution

Answer

Transposing gives $\begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$. Finally, $A^{-1} = \dfrac{1}{10} \begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$.

Now use $X = A^{-1}B$ to solve the system of linear equations:

Your solution

Answer

$$X = \frac{1}{10}\begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 30 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ Then } x_1 = 3,\ x_2 = 1,\ x_3 = 2.$$

Comparing this approach to the use of Cramer's rule for three equations (in subsection 2 of Section 8.1) we can say that the two methods are both rather tedious!

Equations with no unique solution

If $|A| = 0$, A^{-1} does not exist and therefore it is easy to see that the system of equations has no unique solution. But it is not obvious whether this is because the equations are inconsistent and have no solution or whether they are consistent and have infinitely many solutions.

Task

Consider the systems

(a)
$$\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 - 2x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= 7 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 - 2x_3 &= 3 \\ x_1 - 11x_2 + 9x_3 &= 13 \end{aligned}$$

In system (a) add the first equation to the second. What does this tell you about the system?

Your solution

Answer

The sum is $3x_1 + 2x_2 - x_3 = 7$, which is identical to the third equation. Thus, essentially, there are only two equations $x_1 - x_2 + x_3 = 4$ and $3x_1 + 2x_2 - x_3 = 7$. Now adding these two gives $4x_1 + x_2 = 11$ or $x_2 = 11 - 4x_1$ and then

$$x_3 = 4 - x_1 + x_2 = 4 - x_1 + 11 - 4x_1 = 15 - 5x_1$$

Hence if we give x_1 a value, t say, then $x_2 = 11 - 4t$ and $x_3 = 15 - 5t$. Thus there is an infinite number of solutions (one for each value of t).

In system (b) take the combination 5 times the first equation minus 2 times the second equation. What does this tell you about the system?

Your solution

Answer
The combination is $x_1 - 11x_2 + 9x_3 = 14$, which conflicts with the third equation. There is thus no solution.

In practice, systems containing three or more linear equations are best solved by the method which we shall introduce in Section 8.3.

Exercises

1. Solve the following using the inverse matrix method:

(a) $\begin{array}{rcrcl} 2x & - & 3y & = & 1 \\ 4x & + & 4y & = & 2 \end{array}$ (b) $\begin{array}{rcrcl} 2x & - & 5y & = & 2 \\ -4x & + & 10y & = & 1 \end{array}$ (c) $\begin{array}{rcrcl} 6x & - & y & = & 0 \\ 2x & - & 4y & = & 1 \end{array}$

2. Solve the following equations using matrix methods:

(a) $\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & 0 \end{array}$ (b) $\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 1 \\ -x_1 & & & + & x_3 & = & 1 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$

Answers

1. (a) $x = \dfrac{1}{2},\ y = 0$ (b) A^{-1} does not exist. (c) $x = -\dfrac{1}{22},\ y = -\dfrac{3}{11}$

2. (a) $x_1 = \dfrac{8}{3},\ x_2 = -4,\ x_3 = \dfrac{4}{3}$ (b) $x_1 = \dfrac{1}{2},\ x_2 = 1,\ x_3 = \dfrac{3}{2}$

Solution by Gauss Elimination

8.3

Introduction

Engineers often need to solve large systems of linear equations; for example in determining the forces in a large framework or finding currents in a complicated electrical circuit. The method of Gauss elimination provides a systematic approach to their solution.

Prerequisites

Before starting this Section you should ...

- be familiar with matrix algebra

Learning Outcomes

On completion you should be able to ...

- identify the row operations which allow the reduction of a system of linear equations to upper triangular form
- use back-substitution to solve a system of equations in echelon form
- understand and use the method of Gauss elimination to solve a system of three simultaneous linear equations

1. Solving three equations in three unknowns

The easiest set of three simultaneous linear equations to solve is of the following type:

$$3x_1 = 6,$$

$$2x_2 = 5,$$

$$4x_3 = 7$$

which obviously has solution $[x_1, x_2, x_3]^T = \left[2, \frac{5}{2}, \frac{7}{4}\right]^T$ or $x_1 = 2, x_2 = \frac{5}{2}, x_3 = \frac{7}{4}$.
In matrix form $AX = B$ the equations are

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$$

where the matrix of coefficients, A, is clearly diagonal.

Task

Solve the equations

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}.$$

Your solution

Answer

$[x_1, x_2, x_3]^T = [4, -2, -2]^T$.

The next easiest system of equations to solve is of the following kind:

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ 2x_2 + x_3 &= 12 \\ 3x_3 &= 6. \end{aligned}$$

The last equation can be solved immediately to give $x_3 = 2$.
Substituting this value of x_3 into the second equation gives

$$2x_2 + 2 = 12 \qquad \text{from which} \qquad 2x_2 = 10 \qquad \text{so that} \qquad x_2 = 5$$

Substituting these values of x_2 and x_3 into the first equation gives

$$3x_1 + 5 - 2 = 0 \qquad \text{from which} \qquad 3x_1 = -3 \qquad \text{so that} \qquad x_1 = -1$$

Hence the solution is $[x_1, x_2, x_3]^T = [-1, 5, 2]^T$.
This process of solution is called **back-substitution**.
In matrix form the system of equations is

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix}.$$

The matrix of coefficients is said to be **upper triangular** because all elements below the leading diagonal are zero. Any system of equations in which the coefficient matrix is triangular (whether upper or lower) will be particularly easy to solve.

Task

Solve the following system of equations by back-substitution.

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}.$$

Write the equations in expanded form:

Your solution

Answer

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 7 \\ 3x_2 - x_3 &= 5 \\ 2x_3 &= 2 \end{aligned}$$

Now find the solution for x_3:

Your solution

$x_3 =$

Answer

The last equation can be solved immediately to give $x_3 = 1$.

Using this value for x_3, obtain x_2 and x_1:

Your solution

$x_2 =$ $\qquad$ $x_1 =$

Answer

$x_2 = 2, \quad x_1 = 3$. Therefore the solution is $x_1 = 3, x_2 = 2$ and $x_3 = 1$.

Although we have worked so far with integers this will not always be the case and fractions will enter the solution process. We must then take care and it is always wise to check that the equations balance using the calculated solution.

2. The general system of three simultaneous linear equations

In the previous subsection we met systems of equations which could be solved by back-substitution alone. In this Section we meet systems which are not so amenable and where preliminary work must be done before back-substitution can be used.

Consider the system

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 14 \\ 2x_1 - x_2 - 3x_3 &= 3 \\ 4x_1 + 5x_2 - x_3 &= 7 \end{aligned}$$

We will use the solution method known as **Gauss elimination**, which has three stages. In the first stage the equations are written in matrix form. In the second stage the matrix equations are replaced by a system of equations having the same solution but which are in **triangular form**. In the final stage the new system is solved by **back-substitution**.

Stage 1: Matrix Formulation

The first step is to write the equations in matrix form:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & -3 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \\ 7 \end{bmatrix}.$$

Then, for conciseness, we combine the matrix of coefficients with the column vector of right-hand sides to produce the **augmented matrix**:

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right]$$

If the general system of equations is written $AX = B$ then the augmented matrix is written $[A|B]$.

Hence the first equation

$$x_1 + 3x_2 + 5x_3 = 14$$

is replaced by the first row of the augmented matrix,

$1 \quad 3 \quad 5 \quad | \quad 14$ and so on.

Stage 1 has now been completed. We will next triangularise the matrix of coefficients by means of **row operations**. There are three possible row operations:

- interchange two rows;
- multiply or divide a row by a non-zero constant factor;
- add to, or subtract from, one row a multiple of another row.

Note that interchanging two rows of the augmented matrix is equivalent to interchanging the two corresponding equations. The shorthand notation we use is introduced by example. To interchange row 1 and row 3 we write $R1 \leftrightarrow R3$. To divide row 2 by 5 we write $R2 \div 5$. To add three times row 1 to row 2, we write $R2 + 3R1$. In the Task which follows you will see where these annotations are placed.

Note that these operations neither create nor destroy solutions so that at every step the system of equations has the same solution as the original system.

Stage 2: Triangularisation

The second stage proceeds by first eliminating x_1 from the second and third equations using row operations.

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right] \begin{array}{l} \\ R2 - 2\times R1 \\ R3 - 4\times R1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right]$$

In the above we have subtracted twice row (equation) 1 from row (equation) 2.
In full these operations would be written, respectively, as

$$(2x_1 - x_2 - 3x_3) - 2(x_1 + 3x_2 + 5x_3) = 3 - 2\times 14 \quad \text{or} \quad -7x_2 - 13x_3 = -25$$

and

$$(4x_1 + 5x_2 - x_3) - 4(x_1 + 3x_2 + 5x_3) = 7 - 4\times 14 \quad \text{or} \quad -7x_2 - 21x_3 = -49.$$

Now since all the elements in rows 2 and 3 are negative we multiply throughout by -1:

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right] \begin{array}{l} \\ R2 \times (-1) \\ R3 \times (-1) \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 7 & 21 & 49 \end{array}\right]$$

Finally, we eliminate x_2 from the third equation by subtracting equation 2 from equation 3
i.e. $R3 - R2$:

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 7 & 21 & 49 \end{array}\right] \begin{array}{l} \\ \\ R3 - R2 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 0 & 8 & 24 \end{array}\right]$$

The system is now in triangular form.

Stage 3: Back Substitution

Here we solve the equations from bottom to top. At each step of the back substitution process we encounter equations which only have a **single** unknown and so can be easily solved.

Task

Now complete the solution to the above system by back-substitution.

Your solution

Answer
In full the equations are

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 14 \\ 7x_2 + 13x_3 &= 25 \\ 8x_3 &= 24 \end{aligned}$$

From the last equation we see that $x_3 = 3$.
Substituting this value into the second equation gives

$$7x_2 + 39 = 25 \quad \text{or} \quad 7x_2 = -14 \quad \text{so that} \quad x_2 = -2.$$

Finally, using these values for x_2 and x_3 in equation 1 gives $x_1 - 6 + 15 = 14$. Hence $x_1 = 5$. The solution is therefore $[x_1, x_2, x_3]^T = [5, -2, 3]^T$

Check that these values satisfy the original system of equations.

Task

Solve

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 2 \\ 4x_1 + x_2 + 2x_3 &= 2 \\ x_1 - x_2 + 3x_3 &= 3 \end{aligned}$$

Write down the augmented matrix for this system and then interchange rows 1 and 3:

Your solution

Answer
Augmented matrix

$$\left[\begin{array}{rrr|r} 2 & -3 & 4 & 2 \\ 4 & 1 & 2 & 2 \\ 1 & -1 & 3 & 3 \end{array}\right] \quad R1 \leftrightarrow R3 \quad \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 4 & 1 & 2 & 2 \\ 2 & -3 & 4 & 2 \end{array}\right]$$

Now subtract suitable multiples of row 1 from row 2 and from row 3 to eliminate the x_1 coefficient from rows 2 and 3:

Your solution

Answer

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 4 & 1 & 2 & 2 \\ 2 & -3 & 4 & 2 \end{array}\right] \quad \begin{array}{l} \\ R2 - 4R1 \\ R3 - 2R1 \end{array} \quad \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 5 & -10 & -10 \\ 0 & -1 & -2 & -4 \end{array}\right]$$

Now divide row 2 by 5 and add a suitable multiple of the result to row 3:

Your solution

Answer

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 5 & -10 & -10 \\ 0 & -1 & -2 & -4 \end{array}\right] \quad R2 \div 5 \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -2 & -4 \end{array}\right] \begin{array}{r} \\ \\ R3+R2 \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -4 & -6 \end{array}\right]$$

Now complete the solution using back-substitution:

Your solution

Answer

The equations in full are

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 3 \\ x_2 - 2x_3 &= -2 \\ -4x_3 &= -6. \end{aligned}$$

The last equation reduces to $x_3 = \frac{3}{2}$.

Using this value in the second equation gives $x_2 - 3 = -2$ so that $x_2 = 1$.

Finally, $x_1 - 1 + \frac{9}{2} = 3$ so that $x_1 = -\frac{1}{2}$.

The solution is therefore $[x_1, x_2, x_3]^T = \left[-\frac{1}{2}, 1, \frac{3}{2}\right]^T$.

You should check these values in the original equations to ensure that the equations balance.
Again we emphasise that we chose a particular set of procedures in Stage 1. This was chosen mainly to keep the arithmetic simple by delaying the introduction of fractions. Sometimes we are courageous and take fewer, harder steps.

An important point to note is that when in Stage 2 we wrote $R2 - 2R1$ against row 2; what we meant is that row 2 is replaced by the combination (row $2) - 2\times$(row 1).
In general, the operation

$$\text{row } i - \alpha \times \text{row } j$$

means replace **row** i by the combination

$$\text{row } i - \alpha \times \text{row } j.$$

3. Equations which have an infinite number of solutions

Consider the following system of equations

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 3 \\ 2x_1 - 3x_2 + 4x_3 &= -4 \\ x_1 - x_2 + x_3 &= -1 \end{aligned}$$

In augmented form we have:

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 2 & -3 & 4 & -4 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

Now performing the usual Gauss elimination operations we have

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 2 & -3 & 4 & -4 \\ 1 & -1 & 1 & -1 \end{array}\right] \begin{array}{c} \\ R2 - 2 \times R1 \\ R3 - R1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 0 & -5 & 10 & -10 \\ 0 & -2 & 4 & -4 \end{array}\right]$$

Now applying $R2 \div -5$ and $R3 \div -2$ gives

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \end{array}\right]$$

Then $R2 - R3$ gives

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that all the elements in the last row are zero. This means that the variable x_3 can take any value whatsoever, so let $x_3 = t$ then using back substitution the second row now implies

$$x_2 = 2 + 2x_3 = 2 + 2t$$

and then the first row implies

$$x_1 = 3 - x_2 + 3x_3 = 3 - (2 + 2t) + 3(t) = 1 + t$$

In this example the system of equations has an infinite number of solutions:

$$x_1 = 1 + t, \qquad x_2 = 2 + 2t, \qquad x_3 = t \quad \text{or} \quad [x_1, x_2, x_3]^T = [1 + t, 2 + 2t, t]^T$$

where t can be assigned any value. For every value of t these expressions for x_1, x_2 and x_3 will simultaneously satisfy each of the three given equations.

Systems of linear equations arise in the modelling of electrical circuits or networks. By breaking down a complicated system into simple loops, Kirchhoff's law can be applied. This leads to a set of linear equations in the unknown quantities (usually currents) which can easily be solved by one of the methods described in this Workbook.

Engineering Example 3

Currents in three loops

In the circuit shown find the currents (i_1, i_2, i_3) in the loops.

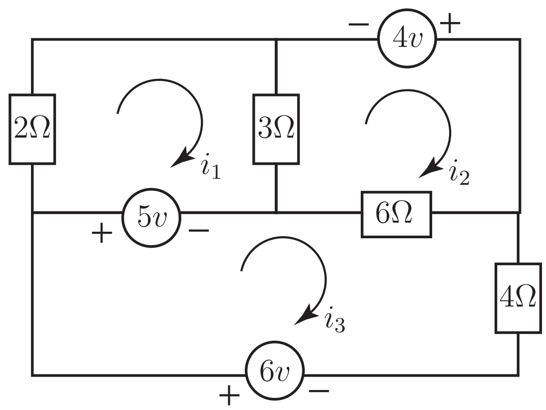

Figure 2

Solution

Loop 1 gives

$$2(i_1) + 3(i_1 - i_2) = 5 \quad \rightarrow \quad 5i_1 - 3i_2 = 5$$

Loop 2 gives

$$6(i_2 - i_3) + 3(i_2 - i_1) = 4 \quad \rightarrow \quad -3i_1 + 9i_2 - 6i_3 = 4$$

Loop 3 gives

$$6(i_3 - i_2) + 4(i_3) = 6 - 5 \quad \rightarrow \quad -6i_2 + 10i_3 = 1$$

Note that in loop 3, the current generated by the $6v$ cell is positive and for the $5v$ cell negative in the direction of the arrow.
In matrix form

$$\begin{bmatrix} 5 & -3 & 0 \\ -3 & 9 & -6 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

Solving gives

$$i_1 = \frac{34}{15}, \quad i_2 = \frac{19}{9}, \quad i_3 = \frac{41}{30}$$

Engineering Example 4

Velocity of a rocket

The upward velocity of a rocket, measured at 3 different times, is shown in the following table

Time, t (seconds)	Velocity, v (metres/second)
5	106.8
8	177.2
12	279.2

The velocity over the time interval $5 \leq t \leq 12$ is approximated by a quadratic expression as

$$v(t) = a_1 t^2 + a_2 t + a_3$$

Find the values of a_1, a_2 and a_3.

Solution

Substituting the values from the table into the quadratic equation for $v(t)$ gives:

$$\begin{aligned} 106.8 &= 25a_1 + 5a_2 + a_3 \\ 177.2 &= 64a_1 + 8a_2 + a_3 \\ 279.2 &= 144a_1 + 12a_2 + a_3 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Applying one of the methods from this Workbook gives the solution as

$$a_1 = 0.2905 \qquad a_2 = 19.6905 \qquad a_3 = 1.0857 \quad \text{to 4 d.p.}$$

As the original values were all **experimental observations** then the values of the unknowns are all **approximations**. The relation $v(t) = 0.2905t^2 + 19.6905t + 1.0857$ can now be used to predict the approximate position of the rocket for any time within the interval $5 \leq t \leq 12$.

Exercises

Solve the following using Gauss elimination:

1.
$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & 0 \end{array}$$

2.
$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 1 \\ -x_1 & & & + & x_3 & = & 1 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$$

3.
$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 3 \\ x_1 & - & 2x_2 & - & x_3 & = & 1 \end{array}$$

4.
$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & -1 \\ 3x_1 & + & x_2 & + & x_3 & = & 4 \\ 11x_1 & - & x_2 & - & 3x_3 & = & 10 \end{array}$$

You may need to think carefully about this system.

Answers

(1) $x_1 = \dfrac{8}{3},\ x_2 = -4,\ x_3 = \dfrac{4}{3}$

(2) $x_1 = \dfrac{1}{2},\ x_2 = 1,\ x_3 = \dfrac{3}{2}$

(3) $x_1 = 2,\ x_2 = 1,\ x_3 = -1$

(4) infinite number of solutions: $x_1 = t,\ x_2 = 11 - 10t,\ x_3 = 7t - 7$